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TECHNICAL NOTE

The penetration rate of solid–liquid phase-change heat transfer interface with different kinds of boundary conditions

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INTRODUCTION

Investigations of one-dimensional moving boundary problems associated with freezing and thawing have been of great practical use and theoretical significance, as in freezing and melting of lake ice, cooling of large masses of igneous rock, materials processing and purification, metal casting and growth of pure crystals from melts and solutions. Many analytical solutions have been reported with a few special classes of boundary and initial conditions [1–3], and different approximate solutions have been obtained to predict the temperature distribution and interface movement in the phase change heat transfer process [4–6] for complicated boundary conditions. However, few reports are available in the literature related to the discussion of the penetration rate of phase change interface. Here, we will present a comparison of penetration rate characteristics of conduction-dominated freezing phase change interface for the case of a semi-infinite body with different boundary conditions. The emphasis focuses on the two-phase problems with convective cooling boundary conditions.

PENETRATION RATE OF SIMPLE FREEZING PROBLEMS

The problems consider a semi-infinite body of phase change medium, as shown in Fig. 1. The initial temperature of the liquid (T_0) is assumed to be uniform and equal to the fusion or freezing temperature, T_f. At $t \geqslant 0$, the fixed boundary is subject to the constant temperature (T_w) or specified energy flow, either a specified heat flux (q) or convective cooling with a constant heat transfer coefficient, h, and a constant sub-freezing ambient temperature, T_a. As freezing takes place, the temperature changes in the solid region. The problems can be solved accurately for different cases accordingly.

1. *Constant surface temperature*, $T_w < T_f$

The solution of the phase change interface movement for the classical Stefan problem [7], where λ is given by the following equation, is:

$$s(t) = 2\lambda(\alpha_s t)^{1/2} \quad (1)$$

$$\lambda e^{\lambda^2}\operatorname{erf}(\lambda) = \frac{C_p(T_f - T_w)}{L\sqrt{\pi}}. \quad (2)$$

Define the penetration rate as:

$$V(t) \equiv \frac{ds}{dt}. \quad (3)$$

We have

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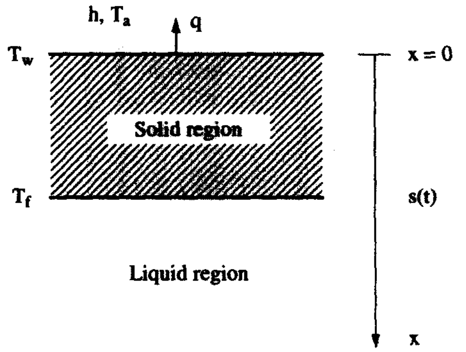

Fig. 1. Physical model for one-dimensional semi-infinite freezing problem.

$$V = \frac{\lambda\sqrt{\alpha_s}}{\sqrt{t}}, \quad V' = -\frac{1}{2}\frac{\lambda\sqrt{\alpha_s}}{t\sqrt{t}} < 0. \quad (4)$$

From equation (4) it is obvious that the penetration rate of phase change interface is decreasing monotonously.

2. *Constant heat flux*, q

The accurate solution for this case can be expressed in the form of Taylor series expansions [8], that is,

$$\xi = \tau - \frac{\tau^2}{2} + \tfrac{5}{6}\tau^3 - \tfrac{17}{8}\tau^4 + \cdots \quad (5)$$

where

$$\xi = \frac{qs}{\rho\alpha_s L}, \quad \tau = \frac{q^2 t}{\rho^2\alpha_s L^2} \quad (6)$$

and, therefore, we have

$$V = \frac{q}{\rho L} - \frac{q^3}{k_s\rho^3 L^3}t + \frac{5}{2}\frac{q^5}{k_s^2\rho^5 L^5}t^2 - \frac{17}{2}\frac{q^7}{k_s^3\rho^7 L^7}t^3 + \cdots. \quad (7)$$

Obviously, the penetration rate reaches the maximum value, $q/\rho L$, at the initiation of freezing and then decreases with time.

3. *Convective cooling boundary condition*

Lozano and Reemsten [9] obtained the solution of the penetration rate for this case, which can be written as:

NOMENCLATURE

C_p	heat capacity
h	convective heat transfer coefficient
k	thermal conductivity
L	latent heat of fusion
q	heat flux
s	phase change interface position
Ste	Stefan number, $C_s(T_f - T_\infty)/L$
Ste'	modified Stefan number, $C_l(T_0 - T_f)/L$
t	time
t_m	coefficient, defined in equation (16)
T	temperature
V	penetration rate.

Greek symbols

α	thermal diffusivity
δ	thickness of the thermal boundary layer
ρ	density.

Subscripts

0	initial
f	freezing
l	liquid phase
s	solid phase
w	wall
a	ambient.

$$s = \frac{h(T_f - T_a)}{\rho L} t - \frac{h^3(T_f - T_a)^2(\alpha_s L\rho + k_s(T_f - T_a))}{2k_s\alpha_s L^3\rho^3} t^2 + \cdots \tag{8}$$

and, hence, we have

$$V = \frac{h(T_f - T_a)}{\rho L} - \frac{h^3(T_f - T_a)^2(\alpha_s L\rho + k_s(T_f - T_a))}{k_s\alpha_s L^3\rho^3} t + \cdots \tag{9}$$

The maximum penetration rate will be $h(T_f - T_a)/\rho L$, which takes place at the initiation of freezing, $t = 0$.

It can be concluded from the above three cases that, for the simple freezing heat transfer process, the maximum penetration rate exists at the initiation of phase change and then decreases with time. Likewise, for the isothermal boundary condition, penetration rate decreases with the inverse ratio of $\sqrt{t}$; however, for the boundary conditions of specified energy flow, the penetration rate proceeds almost linearly with time when freezing time, t, is small.

PENETRATION RATE OF TWO-PHASE FREEZING PROBLEMS

For the cases where the initial temperature of the liquid (T_0) is assumed to be uniform and higher than the fusion temperature, T_f, the temperature will change in both solid and liquid regions. The statements of the boundary conditions are the same as the single-phase cases.

For such a two-phase problem with isothermal boundary condition, the solution can be expressed in the same way as equation (1), but λ would be given by

$$\frac{e^{-\lambda^2}}{\operatorname{erf}(\lambda)} + \frac{k_l}{k_s}\left(\frac{\alpha_s}{\alpha_l}\right)^{1/2}\frac{T_f - T_0}{T_f - T_w}\frac{e^{-\lambda(\alpha_s/\alpha_l)}}{\operatorname{erfc}[\lambda(\alpha_s/\alpha_l)^{1/2}]} = \frac{\lambda L\sqrt{\pi}}{C_{p_s}(T_f - T_w)}. \tag{10}$$

That is, the penetration rate decreases with time monotonously.

As to the two-phase problems with the specified energy flow, as discussed by Wang and Ma [10], freezing begins only after a definite period of precooling for the temperature at a cooled surface to reach freezing point, i.e. the freezing process will experience the durations of precooling and freezing. Then there have been no exact solutions until now. Here, the convective boundary condition is discussed as an example, a heat balance integral approximate solution is obtained for $t > t_f$, as [10]:

$$\frac{ds}{dt} = \frac{1}{\rho L}\left(k_s b + \frac{2k_l}{\delta_l}(T_f - T_0)\right) \tag{11}$$

$$\frac{d\delta_l}{dt} = \frac{6\alpha_l}{\delta_l} - 3\frac{ds}{dt} \tag{12}$$

where t_f stands for the precooling time and can be obtained also from the integral solution as [10]

$$t_f = \frac{2k^2}{3\alpha h^2}\left[\ln\frac{T_f - T_a}{T_0 - T_a} + \frac{1}{2}\frac{(T_0 - T_a)^2}{(T_f - T_a)^2} - \frac{1}{2}\right] \tag{13}$$

b is given by

$$b = \frac{T_f - T_a}{s\left(1 + \frac{\alpha_{21}}{2\delta_l}s\right) + \frac{k}{h}\left(1 + \frac{\alpha_{21}}{\delta_l}s\right)} \tag{14}$$

and $\alpha_{21} = (\alpha_l\rho_l)/(\alpha_s\rho_s)$.

Equations (11) and (12) can be easily solved by the Runge–Kutta method. The solutions of the penetration rate for water have been shown in Fig. 2. When the modified Stefan number, Ste', is equal to zero, the results (by the straight line in Fig. 2) stand for the simple freezing with convective cooling boundary case. It is easy to know from Fig. 2 that the penetration rate for the two-phase cases with the convective boundary condition is zero at the initiation of freezing occurrence, which is quite different from that of simple freezing problems. As time goes on, the penetration rate increases to a certain maximum value and then decreases gradually. There exists a transition point of penetration rate in the freezing heat transfer process. The maximum penetration rate takes place not at the beginning and not at the wall surface, as compared with the simple cases.

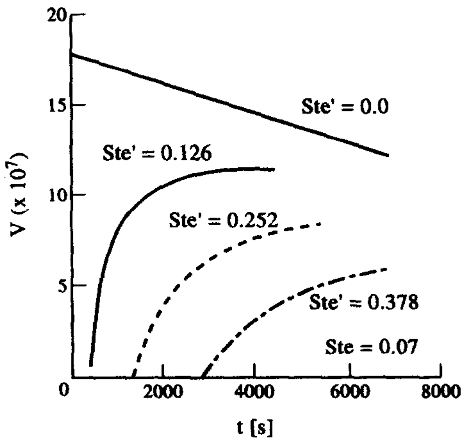

Fig. 2. Penetration rate for water with convective cooling boundary condition.

The comparisons of penetration rate with different *Ste* and *Ste'* are shown in Fig. 3. *Ste* has a more obvious effect on the penetration rate than the *Ste'* number. With the increasing of *Ste* or decreasing of *Ste'*, the time for the attachment of maximum penetration rate will be shorter after the initiation of freezing occurrence.

MATHEMATICAL PREDICTION OF THE MAXIMUM PENETRATION RATE WITH CONVECTIVE COOLING BOUNDARY CONDITION

In order to demonstrate and predict the above analyses, the Chan's quasi-steady analytical solution of the two-phase freezing problem subject to convective cooling boundary condition [11] is introduced. We have:

$$V = \frac{ds}{dt} = \frac{m_1}{s+\frac{k_s}{h}} - \frac{m_2}{\sqrt{t-t_m}} \tag{15}$$

where

$$m_1 = \frac{k_s(T_f-T_a)}{\rho_s L} > 0 \quad m_2 = \frac{k_l(T_0-T_f)}{\rho_s L\sqrt{\pi\alpha_l}} > 0$$

$$t_m = t_f - t_* \quad t_* = \frac{1}{\pi\alpha_l}\left(\frac{k_l}{h}\right)^2\left(\frac{T_0-T_f}{T_f-T_a}\right)^2. \tag{16}$$

Equation (15) was solved by the Runge–Kutta method and the solutions agree well with the above-mentioned heat balance integral approximation solutions [12], as shown in Fig. 4.

Surely penetration rate will be always higher than zero, and so we get from equation (15),

$$\frac{m_2}{m_1}\cdot\frac{s+\frac{k_s}{h}}{\sqrt{t-t_m}} < 1. \tag{17}$$

Meanwhile, the first derivative of the penetration rate is

$$V' = -\frac{m_1 V}{\left(s+\frac{k_s}{h}\right)^2} + \frac{m_2}{2(t-t_m)^{3/2}}. \tag{18}$$

Let $V' = 0$, then

$$\frac{V}{\left(s+\frac{k_s}{h}\right)^2} = \frac{m_2}{2m_1}\frac{1}{(t-t_m)^{3/2}}. \tag{19}$$

With equation (15) substituted into equation (19), it yields

$$V\left(V+\frac{m_2}{\sqrt{t-t_m}}\right)^2 = \frac{m_2 m_1}{2}\frac{1}{(t-t_m)^{3/2}}. \tag{20}$$

The second derivative of the penetration rate is easily derived from equation (20). Introducing equations (17) and (19), it leads to

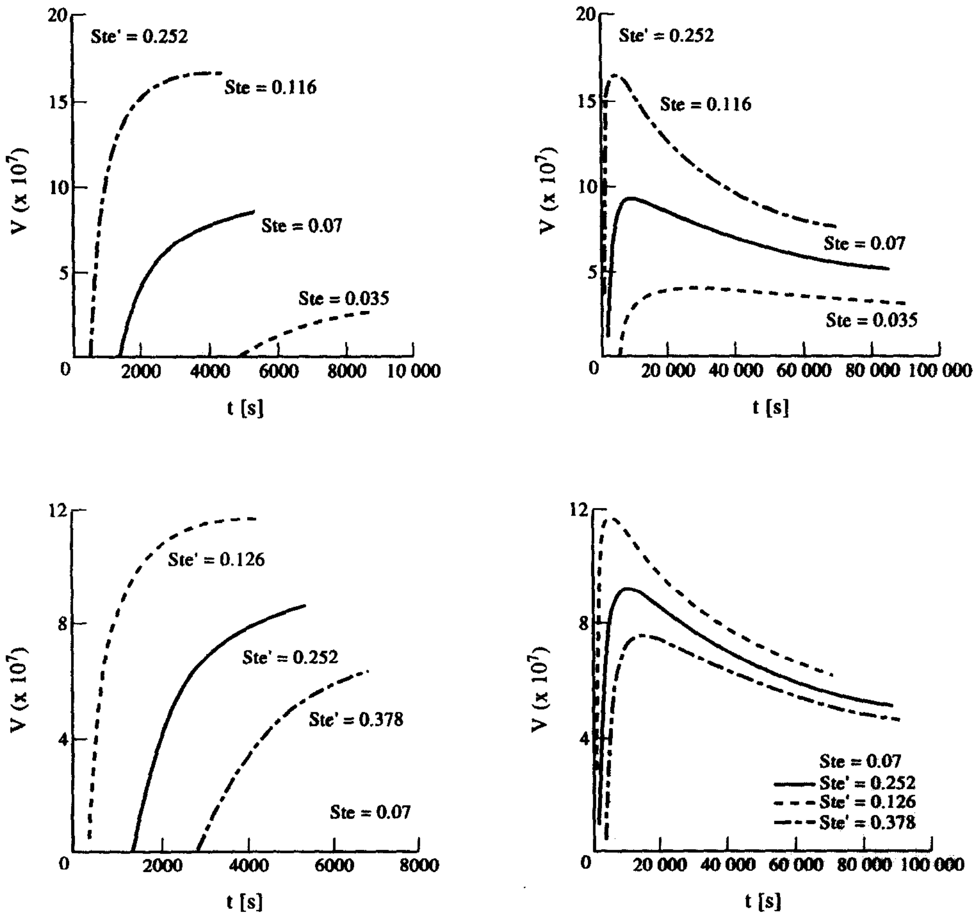

Fig. 3. Comparisons of penetration rate with different *Ste* and *Ste'*.

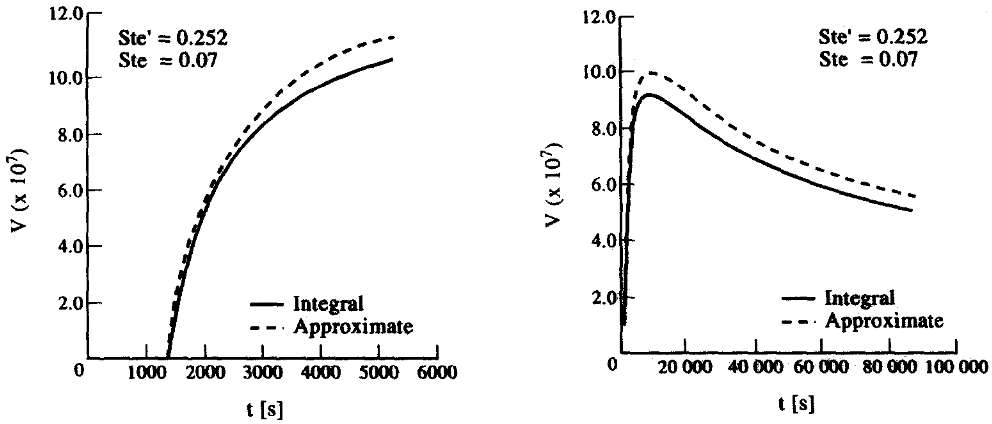

Fig. 4. Comparisons of Chan's quasi-steady analytical solutions and heat balance integral solution.

$$V'' = \frac{m_2}{(t-t_m)^{5/2}}\left[\frac{1}{2}\cdot\frac{m_2}{m_1}\cdot\frac{s+\frac{k_s}{h}}{\sqrt{t-t_m}}-\frac{3}{4}\right]$$

$$< \frac{m_2}{(t-t_m)^{5/2}}\left[\tfrac{1}{2}\cdot 1-\tfrac{3}{4}\right] = -\frac{1}{4}\,\frac{m_2}{(t-t_m)^{5/2}} < 0. \quad (21)$$

That is, an extreme value exists in the system. Let

$$A_1 = \frac{m_2}{\sqrt{t-t_m}} \quad A_2 = \frac{m_2 m_1}{2}\,\frac{1}{(t-t_m)^{3/2}} \text{ and } V = y-\frac{2A_1}{3}. \quad (22)$$

Equation (20) would be simplified to

$$y^3+\xi y+\eta = 0 \quad \text{with} \quad \xi = -\tfrac{1}{3}A_1^2 \quad \eta = -\tfrac{2}{27}A_1^3-A_2. \quad (23)$$

The discriminant equation of equation (23) is $\Delta = (\eta/2)^2+(\xi/3)^3 > 0$, thus there is only one real root for equation (23), i.e. the extreme value of penetration rate is the maximum value.

Solving equation (23) gets

$$V_{\max} = \sqrt[3]{-\frac{\eta}{2}+\sqrt{\left(\frac{\eta}{2}\right)^2+\left(\frac{\xi}{3}\right)^3}} + \sqrt[3]{-\frac{\eta}{2}-\sqrt{\left(\frac{\eta}{2}\right)^2+\left(\frac{\xi}{3}\right)^3}}+\xi. \quad (24)$$

Combining with equation (15), equation (24) can be used to predict the maximum penetration rate of two-phase freezing problems with convective cooling boundary condition. When $Ste = 0.07$ and $Ste' = 0.252$ for water, the prediction result of $V_{\max}$ from equations (15) and (24) is 1.001 μm s^{-1}, while the solution from the integral approximate equations of (11) and (12) is 0.9216 μm s^{-1}.

CONCLUSIONS

For the one-dimensional conduction-dominated simple freezing process, the maximum penetration rate takes place at the initiation of phase change and then decreases with time. Likewise, for the isothermal boundary condition, penetration rate decreases with the inverse ratio of $\sqrt{t}$; however, for the boundary conditions of specified energy flow, the penetration rate proceeds almost linearly with time when freezing time, t, is small.

For the two-phase freezing problems with isothermal boundary condition, the characteristic of penetration rate is similar to that of the simple freezing case. But the maximum penetration rate was found to occur at a certain distance away from the cold wall, not at the initiation of freezing for the specified energy flow boundary condition. The results were further demonstrated mathematically and the equation for predicting the maximum penetration rate was derived, which may be of great practical significance.

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